

Inviscid Burgers Equation, Painlevé Analysis and a Bäcklund Transformation

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The inviscid Burgers equation is studied with the Painlevé analysis. A Bäcklund transformation is constructed. Then we give the symmetry generators. A two-dimensional case is also investigated.

The particular Monge-Ampère equation

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + \sigma \frac{\partial^2 u}{\partial x^2}, \quad (1)$$

known as Burgers equation, represents the simplest equation which incorporates both amplitude and diffusion effects. The inviscid Burgers equation is given by

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x}, \quad (2)$$

i.e. we set $\sigma=0$ in the Burgers equation. Thus the diffusion effects are neglected. It is well known that the inviscid Burgers equation admits the implicit solution [1]

$$u(x, t) = f(x + t u(x, t)), \quad (3)$$

where f is a smooth function. From (3) we have

$$\frac{\partial u}{\partial t} = \left(u + t \frac{\partial u}{\partial t}\right) f' = \frac{u}{1 - t f'} f', \quad (4a)$$

$$\frac{\partial u}{\partial x} = \left(1 + t \frac{\partial u}{\partial x}\right) f' = \frac{1}{1 - t f'} f', \quad (4b)$$

whence (2) follows. Obviously we would call (2) integrable. The solution exists as long as $1 - t f' \neq 0$. Hence we can expect that for some x there will be a first t value such that $1 - t f' = 0$. At this point there will be a shock. In transport processes this corresponds to a critical length.

Let us now apply the Painlevé test to (2). The Painlevé test has been described in detail in literature (compare [2, 3, 4] and references therein).

The Painlevé property for ordinary differential equations, namely that all movable singularities in the solutions are poles, has been known since the famous

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work of Sonya Kovalevskaya, to be related to integrability. In recent years, it has been generalized so as to apply to (nonlinear) partial differential equations. The conjecture is that a partial differential equation is integrable (by the inverse scattering transform or Bäcklund transform, or some other method), if and only if it possesses the Painlevé property (possibly after a change of variables) (see [2, 3, 4] for more details).

The formulation is as follows: Suppose we have a nonlinear partial differential equation with analytic coefficients (or a set of N partial differential equations for N functions, denoted collectively by u). Complexify the functions, so that we now have N holomorphic functions of the complex independent variables. Let ϕ be a holomorphic function such that $\phi=0$ is not a characteristic hypersurface for the partial differential equation. Let us consider the powers series

$$u = \phi^{-n} \sum_{j=0}^{\infty} u_j \phi^j \quad (5)$$

with n being a suitable positive integer. Now we substitute this power series into the nonlinear partial differential equation. This leads to recursion relations between the u_n . The Painlevé property states that these recursion relations should be consistent, and that the expansion (5) should contain the maximal number of arbitrary functions (counting ϕ as one of them). We also say that the partial differential equation passes the Painlevé test. In the so-called weak Painlevé test we allow n and j to be rational numbers in the expansion (5) (see [2] for more details).

In the present case we find from the Painlevé test that (2) admits an expansion of the form

$$u = u_0 + u_1 \phi^n + u_2 \phi^{2n} + \dots, \quad (6)$$

where

$$n = 1/2, \quad (7)$$

and ϕ and u_j are smooth functions of x and t . For the expansion coefficients u_j we obtain the relations collecting terms with the same power of ϕ . We find an infinite coupled system for the expansion coefficients u_j . For the first two expansion coefficients, i.e. u_0, u_1 , we find

$$\phi^{-1/2}: u_0 \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial t}, \quad (8a)$$

$$\phi^0: \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + \frac{1}{2} u_1^2 \frac{\partial \phi}{\partial x} = 0. \quad (8b)$$

If we require that u_0 satisfies the inviscid Burgers equation

$$\frac{\partial u_0}{\partial t} = u_0 \frac{\partial u_0}{\partial x}, \quad (9)$$

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then we find from the infinite coupled system for the expansion coefficient that

$$u_1 = u_2 = \dots = 0. \quad (10)$$

Inserting (8a) into (9) gives

$$\left(\frac{\partial \phi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial t^2} + \left(\frac{\partial \phi}{\partial t}\right)^2 \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial t} \frac{\partial^2 \phi}{\partial x \partial t} = 0. \quad (11)$$

Thus from (8a) and (9) it follows that

$$u = \frac{(\partial \phi)/(\partial t)}{(\partial \phi)/(\partial x)}, \quad (12)$$

where ϕ satisfies (11) and u satisfies (2). Thus we have a Bäcklund transformation. Equation (11) can be linearized applying the Legendre transformation [1, 2]. Equation (11) also arises in many non-Painlevé equations at the resonance [2].

Under the transformation

$$\bar{t}(x, t) = t, \quad (13a)$$

$$\bar{x}(x, t) = u(x, t), \quad (13b)$$

$$\bar{u}(\bar{x}(x, t), \bar{t}(x, t)) = x, \quad (13c)$$

(2) takes the form

$$\frac{\partial \bar{u}}{\partial \bar{t}} = -\bar{x}. \quad (14)$$

The general solution of (14) is given by

$$\bar{u}(\bar{x}, \bar{t}) = g(\bar{x}) - \bar{x} \bar{t}, \quad (15)$$

where g is an arbitrary smooth function. Inserting the transformation (13) gives

$$t u(x, t) + x = g(u(x, t)). \quad (16)$$

On inspection we see that (2) admits the symmetry generators

$$T = \frac{\partial}{\partial t}, \quad X = \frac{\partial}{\partial x}, \quad S = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \quad (17)$$

$$U = -2t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}.$$

The group theoretical reduction of (2) applying the symmetry generator S yields the solution

$$u(x, t) = -x/t. \quad (18)$$

The group theoretical reduction of (2) applying the symmetry generator U gives the solution

$$u(x, t) = -\frac{x}{2t} \pm \frac{1}{2} \sqrt{\frac{x^2}{t^2} - \frac{4C}{t}}, \quad (19)$$

where C is a constant of integration.

The two dimensional case of the inviscid Burgers equation is given by

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y}. \quad (20)$$

We find the implicit solution

$$u(x, y, t) = f(x + u(x, y, t)t, y + u(x, y, t)t). \quad (21)$$

Again we find an expansion of the form

$$u = u_0 + u_1 \phi^{1/2} + u_2 \phi + \dots, \quad (22)$$

where

$$u_0 = \frac{\frac{\partial \phi}{\partial t}}{\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}} \quad (23)$$

and

$$\frac{\partial u_0}{\partial t} - u_0 \frac{\partial u_0}{\partial x} - u_0 \frac{\partial u_0}{\partial y} - \frac{1}{2} u_1^2 \frac{\partial \phi}{\partial x} - \frac{1}{2} u_1^2 \frac{\partial \phi}{\partial y} = 0. \quad (24)$$

Again we can set $u_1 = u_2 = \dots = 0$. Then the quantity ϕ satisfies the equation

$$\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right)^2 - 2 \frac{\partial^2 \phi}{\partial t \partial x} \frac{\partial \phi}{\partial t} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) \\ - 2 \frac{\partial^2 \phi}{\partial t \partial y} \frac{\partial \phi}{\partial t} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) \\ + \left(\frac{\partial \phi}{\partial t} \right)^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + 2 \frac{\partial^2 \phi}{\partial x \partial y} \right) = 0, \end{aligned} \quad (25)$$

and (24) takes the form

$$\frac{\partial u_0}{\partial t} - u_0 \frac{\partial u_0}{\partial x} - u_0 \frac{\partial u_0}{\partial y} = 0. \quad (26)$$

To summarize: We find that the inviscid Burgers equation (2) does not pass the Painlevé test, whereas the Burgers equation (1) with $\sigma \neq 0$ passes the test. We find an expansion of the form

$$u = \phi^{-1} \sum_{j=0}^{\infty} u_j \phi^j \quad (27)$$

for the Burgers equation, where ϕ and u_2 are arbitrary. Although the inviscid Burgers equation does not pass the Painlevé test (only the weak Painlevé test), we find a Bäcklund transformation for the inviscid Burgers equation. For the Burgers equation (1) the Painlevé test provides an auto-Bäcklund transformation [2] and a Bäcklund transformation (so-called Cole Hopf transformation [1]) to the linear diffusion equation [2].

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